Counting

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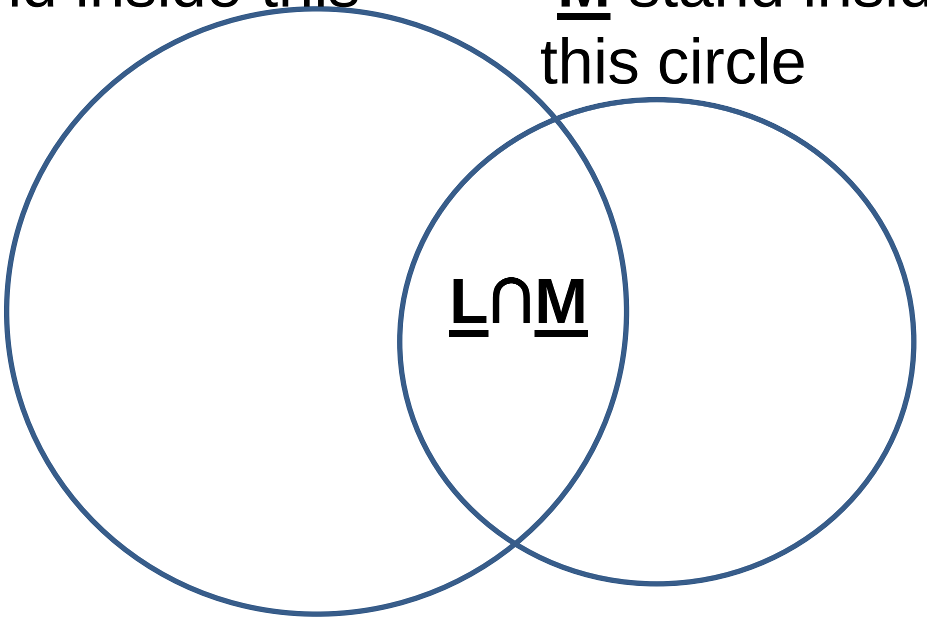
First I want to explain the **principle of inclusion-exclusion**

The Liverpool fans **L** stand inside this circle

The maths fans **M** stand inside this circle



L U **M** = the whole group

How many people in the group?

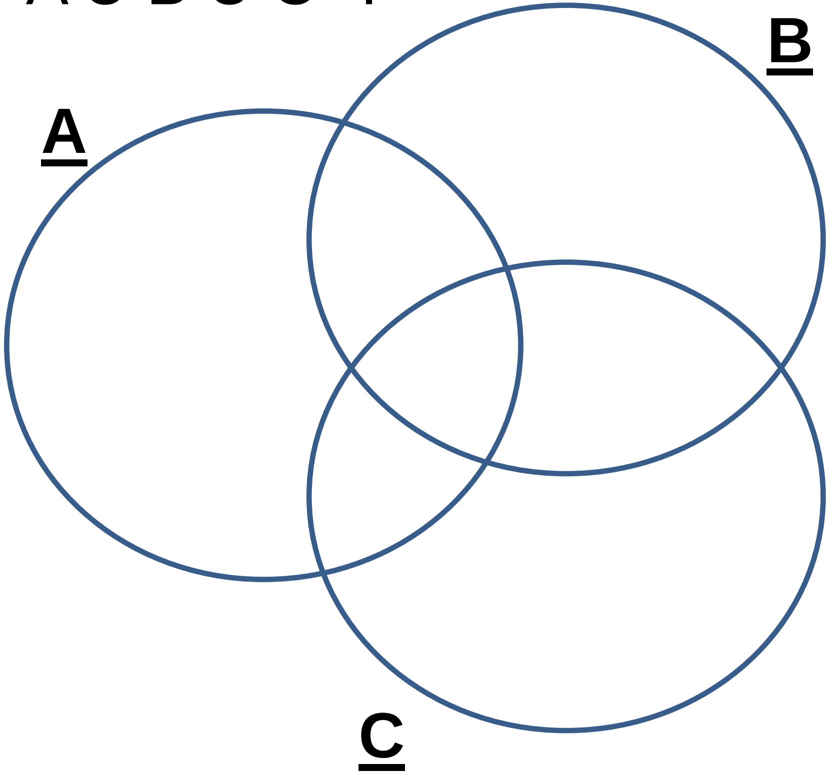
L U M = L + M - (L∩M) = 20+10-5=25

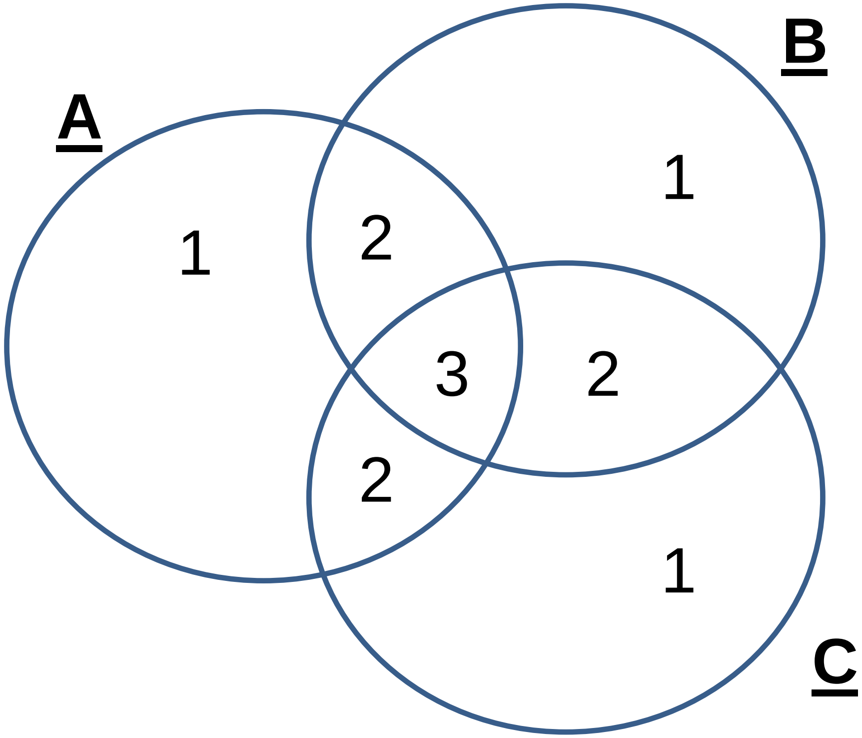
Suppose in a group of young people, 20 support Liverpool, 10 think maths is cool and 5 **both** support Liverpool **and** think that maths is cool.

L = 20, M = 10, L∩M=5

So L+M is **corrected** by subtracting L∩M

How does this work with three overlapping sets? If we just add up A + B + C how do we correct this to get the number in A U B U C ?

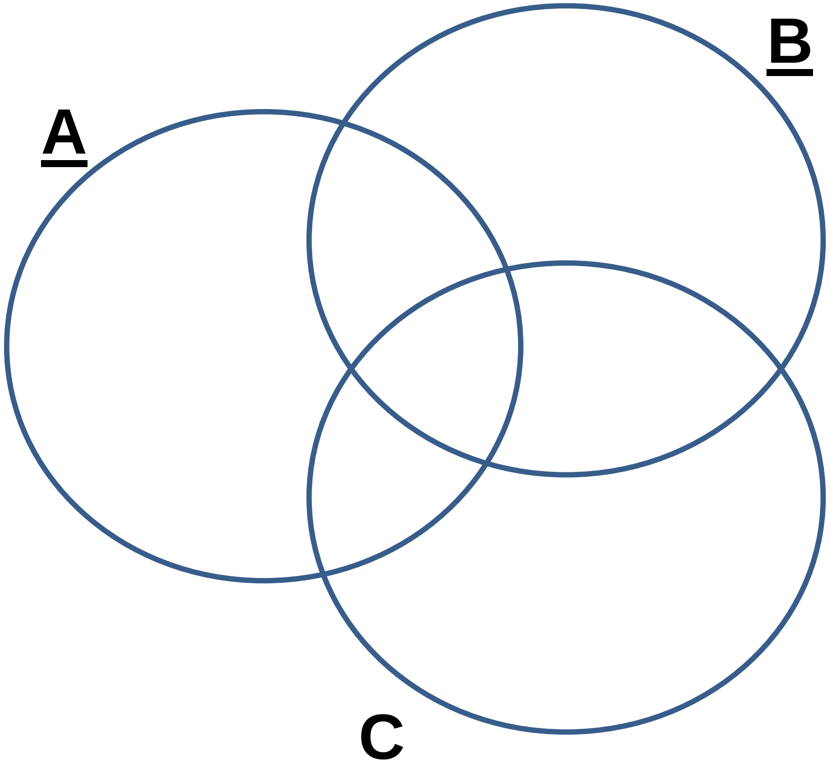




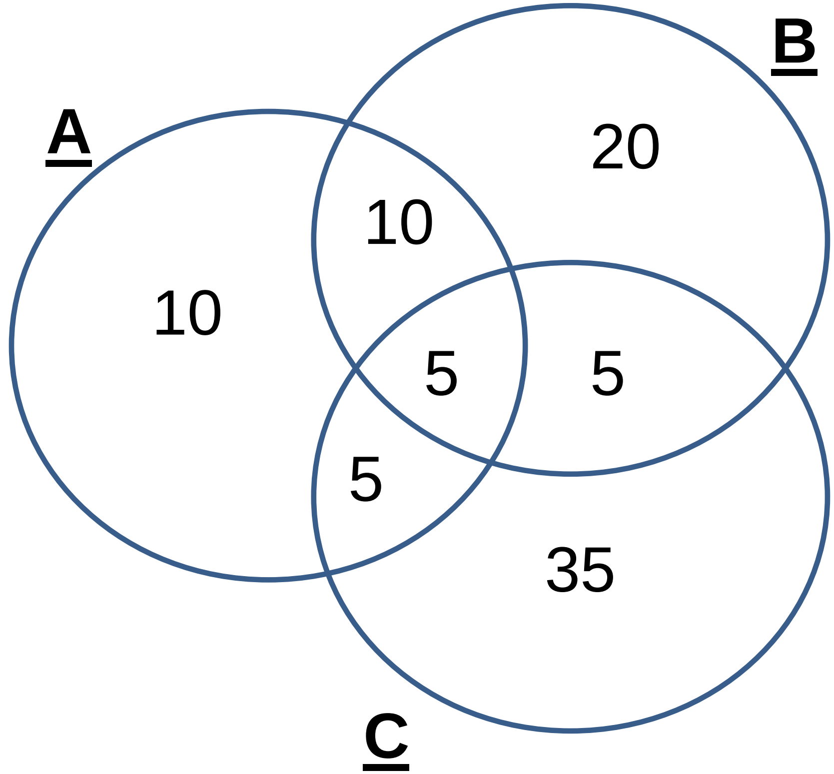
These are the number of times someone is counted by just adding up A + B + C

A U B U C = A + B + C – (A ∩ B) – (B ∩ C) – (A ∩ C) + (A∩B∩C)

Example:



These are the actual numbers of people in the various regions



A U B U C = A + B + C – (A ∩ B) – (B ∩ C) – (A ∩ C) + (A∩B∩C)

30 +40+50 – 15 – 10 – 10 + 5

= 90

$$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$$

The total number in all the sets (A U B U C) is:

the sum of the numbers in the sets taken 1 at a time

minus

the sum of the numbers in the intersections taken two at a time

plus

the sum of the numbers in the intersections taken three at a time

The general formula involving *k* sets is called the **principle of inclusion-exclusion**

Problems.....for later; they are on the handout

$$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$$

$$30 + 40 + 50 - 15 - 10 - 10 + x$$

What possible values could *x* have?

$$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$$

$$17 + 30 + 40 - 15 - 10 - 10 + x$$

What possible values could *x* have?

$$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$$

Is this possible?

$$30 + 40 + 50 - 20 - 15 - 20 + x$$

Here’s a different problem, which will turn out to be the same! Consider “maps” between two sets.

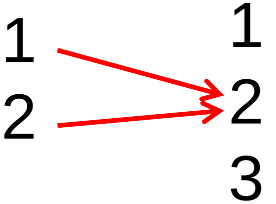
Define **n** to be the numbers 1,2,3,…,n
And **k** to be the numbers 1,2,3,…,k

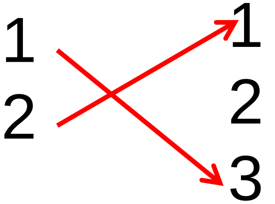
A “map” from **n** to **k** means a rule *f* which takes each number in the source **n** to a definite number in the target **k.**
So there is one arrow from each number on the left, hitting some number on the right

f can take 1 to 1,2 or 3 and 2 to 1,2 or 3.
How many possibilities are there altogether?

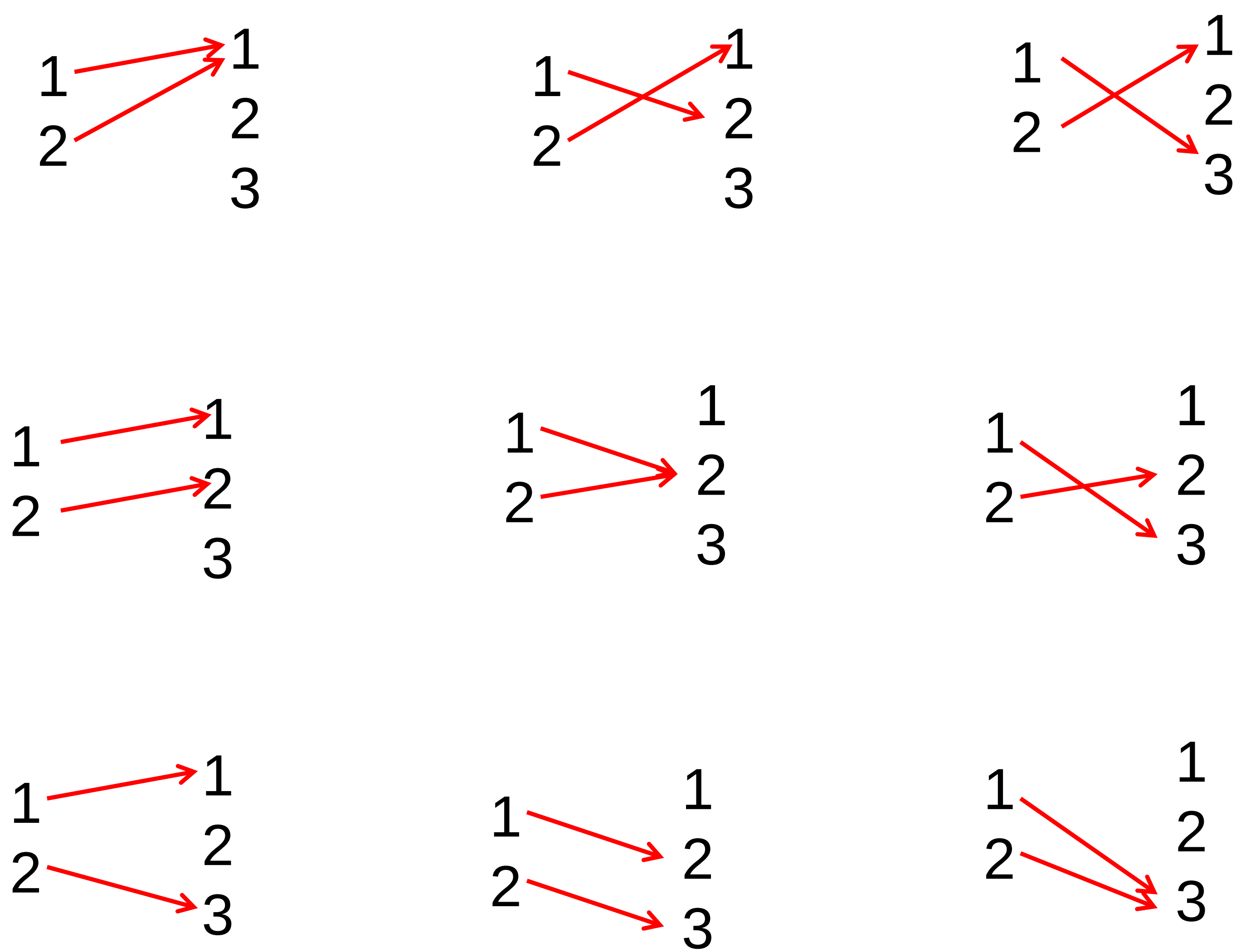
Examples

f: **2** $\rightarrow$ **3**





maps *f:* **2** → **3**



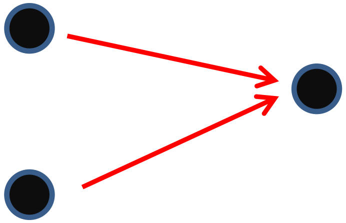
Every number in the source must have an arrow going to a definite place in the target.

Total is $3^2=9$ What is the general result for **n** and **k** ?

Answer: number of maps *f:* n → k *is* k^n

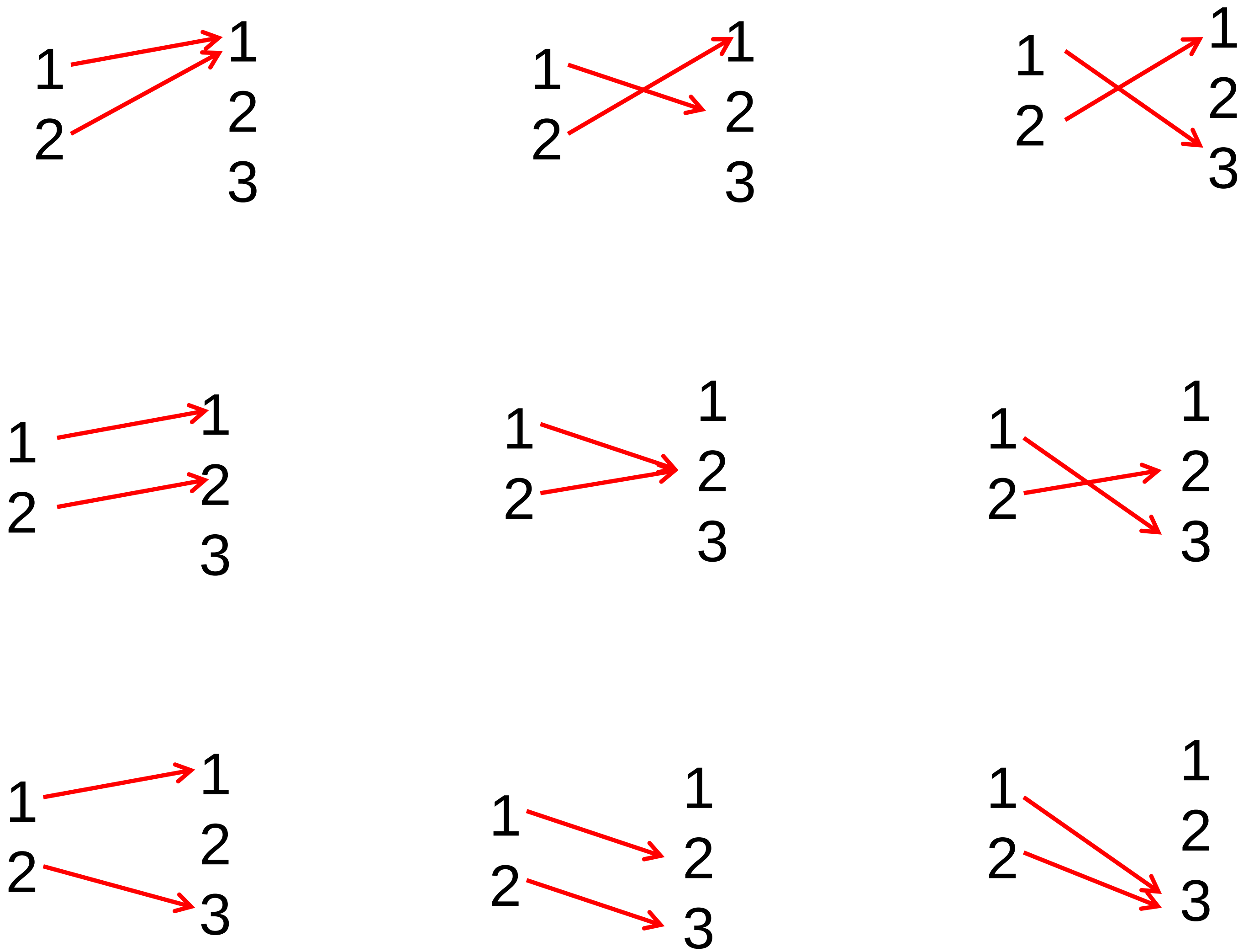
A map is called **injective** if different *x* and *y* always give different values *f*(*x*) and *f*(*y*), so two arrows from the source cannot end up at the same number in the target.

So you **never** get



for an injective map

How many **injective maps** are there between <u>**2**</u> and <u>**3**</u> ?



All but three of these are injective, so 6 out of 9

How many **injective maps** are there between **n** and **k** ?

Maybe you can see that we need $n \leq k$ for this to work.

There are k places where 1 can go,
k-1 places where 2 can go [can't go to same place as 1]
k-2 places where 3 can go [can't go to same place as 1,2]
....
k-n+1 places where n can go
So $k(k-1)(k-2)....(k-n+1)$ possibilities altogether:

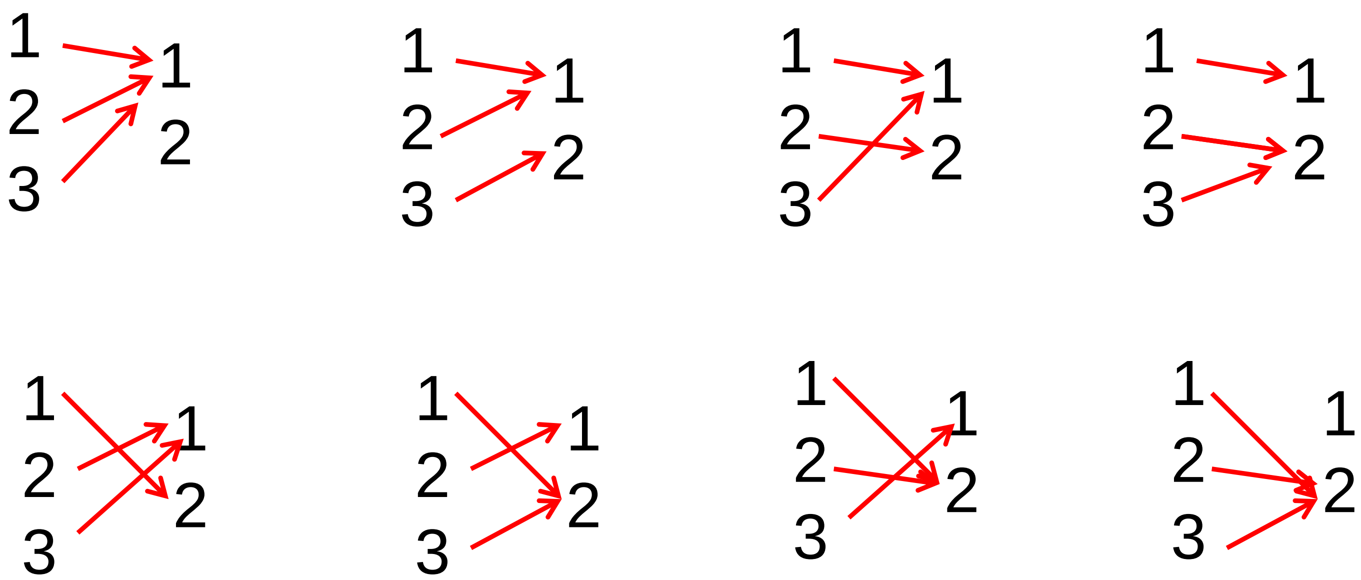
$$\frac{k!}{(k-n)!}$$

E.g. $n = 2$, $k = 4$ gives 12 injective maps **2** → **4** (all except 4 of the 16 maps **2** → **4**)

A map is called **surjective** if every number in the target is hit by at least one arrow.

For $f : \underline{\mathbf{n}} \to \underline{\mathbf{k}}$ this requires $n \geq k$

How many surjective maps are there from **3** to **2** ?



These are all the $2^3 = 8$ maps from **3** to **2**

Maybe you can spot the answer for maps from **4** to **2** or indeed for maps **n** to **2**: how many are surjective?

But however could we count the surjective maps **3** → **3** ?
There are 27 maps altogether. Do we have to list them all and check the ones which are surjective?

Here's a hint:

Let **A** be all the maps which do **not** hit the number 1
Let **B** be all the maps which do **not** hit the number 2
Let **C** be all the maps which do **not** hit the number 3

How many maps in **A?** in **B**? in **C**? 2^3=8 in each
How many maps in **A**∩**B**? $1^3 = 1$, also B∩C etc.
How many maps in **A**∩**B**∩**C ?** 0
How many maps in **A**U**B**U**C?** [These are the *not surjective maps*]
Remember
A U B U C = A + B + C – (A ∩ B) – (B ∩ C) – (A ∩ C) + (A∩B∩C)
8 + 8 + 8 – 1 – 1 – 1 + 0 = 21 **not surjective**
so **27** – **21 = 6 surjective**

We can apply the same idea to, say, maps **4** to **3**

Let **A** be all the maps which do **not** hit the number 1
Let **B** be all the maps which do **not** hit the number 2
Let **C** be all the maps which do **not** hit the number 3

How many maps in **A?** in **B**? in **C**?
How many maps in **A**∩**B**?
How many maps in **A**∩**B**∩**C ?**
How many maps in **A**U**B**U**C?** [These are the *not surjective maps*]

A U B U C = A + B + C – (A ∩ B) – (B ∩ C) – (A ∩ C) + (A∩B∩C)

$$2^4 + 2^4 + 2^4 - 1 - 1 - 1 + 0 = 45$$

out of the total of 3^4 = 81 maps are **not surjective**
so 81 – 45 = 36 maps are surjective.

General result, using the general inclusion-exclusion principle:number of surjective maps **n** to **k** is (when $n \geq k$)

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \ldots$$

where the sum in continued for *k* terms (the next one is zero)

If $n = k$ this must equal the number of injective maps (can you see why this is??), namely $k!$, which is a bit surprising....

e.g. $n=k=4$ gives

$$4^4 - 4(3^4) + 6(2^4) - 4(1^4) = 256 - 324 + 96 - 4 = 24 = 4!$$

Try looking up **inclusion-exclusion principle** on Google. It is just a generalization of

A U B U C = A + B + C – (A ∩ B) – (B ∩ C) – (A ∩ C) + (A∩B∩C)

to any number of sets $\underline{\mathbf{A}}_1$, $\underline{\mathbf{A}}_2$, … $\underline{\mathbf{A}}_k$

For four sets **A**, **B**, **C**, **D** the formula is
A U B U C U D = A + B + C +D – (A ∩ B) – (A ∩ C) – (A ∩ D) – (B ∩ C) – (B ∩ D) – (C ∩ D) + (B ∩ C ∩ D) + (A ∩ C ∩ D) + (A ∩ B ∩ D) + (A ∩ B ∩ C) – (A ∩ B ∩ C ∩ D)

(It's harder to draw a Venn diagram to illustrate this!! There need to be $2^4 - 1 = 15$ regions)